R = pipe radius, cm.
Re = real part of quantity following
r, θ, z = cylindrical coordinates
S^{ij} = stress tensor
t, t', τ = time, sec.
u = axial speed, cm./sec.
$\mathbf{X}$ = position vector of material point at time t'; components X^1, X^2, X^3
$\mathbf{x}$ = position vector of material point at time t; components x^1, x^2, x^3
α = defined by Equation (13)
γ = defined by Equation (18)
Δ = amplitude of displacement wave at center line, cm.
δ = displacement, cm.
$\delta(t)$ = Dirac delta function
ζ = defined by Equation (18)
η = viscosity, poise
Θ = normal stress coefficient, g./cm.
$\lambda_o, \ldots \mu_o, \ldots$ = constants in operator equation
ρ = density, g./cc.
ϕ = phase lag, rad.
Ψ = complex viscosity, poise
$\psi(t)$ = relaxation function, dynes/sq. cm.
ω = frequency, sec.$^{-1}$

LITERATURE CITED

1. Alfrey, T., Jr., and E. F. Gurnee, "Rheology," Vol. 1, Chap. 11, pp. 387-429, F. R. Eirich, ed., Academic Press, New York (1956).
2. Broer, L. J. F., *Appl. Sci. Res.*, **A6**, 226 (1956-'57).
3. Ferry, J. D., "Viscoelastic Properties of Polymers," Chap. 5, Wiley, New York (1961).
4. Fredrickson, A. G., *Chem. Eng. Sci.*, **17**, 155 (1962).
5. ———, "Principles and Applications of Rheology," pp. 139-144, Prentice-Hall, Englewood Cliffs, New Jersey (1964).
6. Kapoor, N. N., M.S. thesis, Univ. Minnesota, Minneapolis, Minnesota (1964).
7. ———, E. A. Brumm, and A. G. Fredrickson, in press.
8. Oldroyd, J. G., *Proc. Roy. Soc. (London)*, **A200**, 523 (1950).
9. ———, *Quart. J. Mech. Appl. Math.*, **4**, 271 (1951).
10. ———, D. J. Strawbridge, and B. A. Toms, *Proc. Phys. Soc.*, **B64**, 44 (1951).
11. Toms, B. A., *Rheologica Acta*, **1**, 137 (1958).
12. ———, and D. J. Strawbridge, *Trans. Faraday Soc.*, **49**, 1225 (1953).
13. Vela, S., Ph.D. thesis, Univ. Minnesota, Minneapolis, Minnesota (1964).
14. Walters, K., *Quart. J. Mech. Appl. Math.*, **13**, 444 (1964).
15. Williams, M. C., and R. B. Bird, *Ind. Eng. Chem. Fundamentals*, **3**, 42 (1964).

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Film Instabilities in Two-Phase Flows

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The major types of instabilities which can alter a given two-phase flow pattern or lead to the breakup of a liquid film are delineated, and their physical mechanisms are discussed. Existing stability criteria are correlated to these basic types, and their application and limitations with respect to actual problems are indicated. Basic areas requiring further study are outlined.

A criterion for determining the breakup length of a liquid film is developed which gives an indication of whether a given instability will lead to low frequency and high amplitude pressure and inventory pulsations. Such phenomena may be undesirable.

Two new aspects of importance to the problem of film instability are introduced which have not been previously studied. These are rotation of the fluids and large disturbances in the flow. Their possible influence on the problem is discussed.

Considerable work has been done to study the motion of liquid films that are driven by gas or vapor flows. A good deal of this work was done in connection with film cooling of rocket nozzles, and some has been done in relation to the design and operation of boilers and condensers. It has been found qualitatively that after some length of flow the vapor-liquid interface becomes wavy, and ultimately the liquid waves either are broken up so that liquid is entrained in the vapor or, in tube flow, the liquid film waves sometimes become so large that they essentially join together from opposite sides and form liquid plugs. This sort of behavior changes the basic flow with resulting changes in the pressure drop and heat transfer and can also lead to other undesirable phenomena in tube flows such as pressure and inventory oscillations.

Accordingly a great deal of effort has been exerted to determine the conditions under which the interfacial waviness occurs so that the various associated flow regimes (such as fog) can be identified and objectionable phenomena avoided. Unfortunately, no general stability criteria for two-phase flows with heat and mass transfer have been determined to date. What do exist are criteria that were either empirically established (and, hence, apply only to a specific configuration, fluid, and restricted range of operating conditions) or that were derived on the basis of adiabatic flows with no mass transfer between the phases. Furthermore, even these criteria apply only to a few of the various types of instabilities possible in an

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actual case. It is rather remarkable that this last point does not seem to have been noticed or discussed heretofore. In all the existing studies of the instability of liquid films (be they experimental or theoretical) the authors choose a given configuration or mathematical model, without discussing either the instability mechanism or its relation to other possible ones, and develop a stability criterion; nor is the particular criterion obtained compared with others found for similar situations. Thus, even if one or another of the existing criteria would predict stability for a given problem, one could not be sure of the result because instability might be caused by a mechanism not accounted for by the criterion that was applied.

It is, therefore, the purpose of this present paper to consider the problem of liquid film instability from a more fundamental view in the hope that some of the apparent existing anomalies can be explained and that results of somewhat greater generality can be obtained.

REVIEW OF EXISTING WORK

Most of the existing work on film instabilities is experimental, although some simplified analyses were also made. Much of this work was done for plane flows, and its applicability to annular flows such as occur in tubes is limited.

The striking aspect of the literature on annular flow is that conflicting criteria are suggested for film breakup; also there is disagreement regarding the significance of various physical parameters. It is beyond the scope of the present paper to review all the work done (see reference *1* for such a comprehensive review). For the present purpose consideration is given to some of the more important representative studies, and their conclusions are summarized on Figure 1. The extensive experimental observations made by Dukler(*2, 3*) in a vertical tube with downward annular flow led him to conclude that the liquid Reynolds number was not a significant parameter that determines wave motion or wave height, although he did note an increase in amplitude and/or frequency of the interface waves with an increase in liquid velocity. Furthermore, Dukler found that there were essentially two significant changes in the nature of the flow. The first of these he associated with a basic change in wave structure that occurs when the energy transferred across the interface reaches some critical value, and the second he attributed to a gas velocity effect (and, hence, a gas Reynolds number) which thins the liquid film and thereby decreases the wave amplitude.

Kinney and his co-workers (*4*) found in their experiments in horizontal ducts that the liquid film becomes wavy when a critical liquid flow rate was exceeded. Furthermore, for more viscous liquid films this critical flow rate was higher. One can thus infer that in contradistinction to Dukler's result the liquid Reynolds number is indeed an important parameter. This discrepancy occurs possibly because a wider range of liquid velocities were covered in reference *4* than in references *2* and *3*. Brauer (*5*) was also led to believe from his observations of a single-phase liquid film on the outside of a vertical cylinder that the occurrence of surface waves is associated with the transition of the liquid film from laminar to turbulent flow, that is, it is related to a critical liquid Reynolds number.

More seemingly contradictory results were obtained by Knuth (*6*) who found surface waves for all liquid flow rates in horizontal ducts. For liquid flow rates larger than some critical value he found a second type of surface wave (one with long wave lengths). Thus again, as in Dukler's work, there appears to be two distinct and different interfacial instabilities. The differences between the work of Kinney and his co-workers and that of Knuth was attributed in the literature to differences in the liquid injection methods. Knuth, however, agreed with Kinney that the instability depended on the liquid Reynolds number. Recall, this disagrees with Dukler's conclusion. Furthermore, Kinney and Knuth are in agreement that the gas Reynolds number has a negligible effect on interfacial stability, whereas Laird (*7*) shows that over isolated ranges of operation the gas Reynolds number does influence the interface. Similar inconsistencies can be seen in other work.

Although the existing work seemingly contains contradictions and disagreements, it does indicate that flow characteristics change and that objectionable phenomena can occur in two-phase flows when the gas-liquid interface becomes wavy. It is, therefore, essential to discuss the possible causes of such interfacial waviness from a fundamental viewpoint. In this regard a summary is presented below of an existing body of basic knowledge that does not appear to have been completely used heretofore by investigators of interfacial stability in two-phase flows.

AUTHOR	N_{Re_g}	N_{Re_L}	
DUKLER	+	–	TWO FLOW CHANGES
KINNEY ET AL	–	+	
BRAUER		+	
KNUTH	–	+	WAVES FOR ALL R_{e_L}
LAIRD	+		

Fig. 1. Summary of existing work.

FOUR PRINCIPAL TYPES OF INSTABILITY

The four different types of hydrodynamic instabilities that singly or in combination can lead to flow waviness and changes in the type of flow are referred to as Tollmien-Schlichting,* Kelvin-Helmholtz, Rayleigh-Taylor, and Bénard instabilities. A summary of their basic characteristics is briefly presented in Figure 2.

The Tollmien-Schlichting instability occurs when a fluid is undergoing transition from laminar to turbulent flow as a result of amplification by viscosity of infinitesimal disturbances in the fluid. These disturbances originate either inside or outside the film as has been demonstrated by changing the turbulence level of the flow external to the boundary layer or film. This type of instability in boundary layers of a homogeneous fluid has been extensively studied (*8*), and it has been conclusively established that waviness occurs under certain conditions. The criterion for the onset of this instability is that the Reynolds number exceed a certain value called the *minimum critical Reynolds number*.

The Kelvin-Helmholtz instability is the second possible type to influence two-phase flows. This instability arises

* This name is used here for convenience and is not always associated with this phenomenon.

NAME	DESCRIPTION	FLUID	CRITERION
TOLLMIEN-SCHLICHTING	LAMINAR-TURBULENT	HOMOGENEOUS	REYNOLDS NUMBER
KELVIN-HELMHOLTZ	RELATIVE MOTION AT INTERFACE	INHOMOGENEOUS	RICHARDSON NUMBER
RAYLEIGH TAYLOR	RELATIVE BODY FORCE AND INTERFACE ORIENTATION	INHOMOGENEOUS	DENSITY GADIENT
BÉNARD	RELATIVE BODY FORCE AND DENSITY GRADIENT ORIENTATION	HOMOGENEOUS	RAYLEIGH NUMBER

Fig. 2. Basic stability types.

when the different layers of a stratified heterogeneous fluid are in relative horizontal motion and is due to the interaction between the fluid media at the interface. The mathematical analysis of this instabilty is described in reference 9. Basically, it is also a small perturbation approach, but the viscosities of the fluids are neglected. The Reynolds number, therefore, plays no important role in this type of instability. Instead a parameter $g(\alpha_1 - \alpha_2)/\alpha_1\alpha_2(U_1 - U_2)^2$ for (stratified) fluids with discontinuous velocities and densities appears as the stability criterion, where g is the acceleration of gravity, α_1 and α_2 are the density ratios defined as $\alpha_1 = \rho_1/(\rho_1 + \rho_2)$, and U_1 and U_2 are the velocities of the two fluids. For fluids with continuous velocity and density distributions in the direction normal to the flow, the above parameter is called the *Richardson number J* and is written as $J = -(gd\rho/dz)/\rho(dU/dz)^2$. This represents the ratio of buoyancy and inertia (or shear) forces. For stability, neglecting surface tension, the disturbance wave number must be less than some value of the dimensionless parameter written in either of the above ways.

The most striking aspect of the Kelvin-Helmholtz instability is that it occurs no matter how small the velocity difference or shear of the two fluids. The instability arises by the crinkling of the interface by the shear that is present. The crinkling occurs even for the smallest differences in the velocities of the two fluids; it can occur even if the motions of both fluids are laminar. The source of the Kelvin-Helmholtz instability lies in the energy stored in the kinetic energy of relative motion of different layers. The tendency toward mixing and instability will be greater, the greater the prevailing shear force. The only counteracting forces are inertia and surface tension.

It is important to note from the stability criterion cited above that in a zero-gravity environment the two-phase flow would be unstable to all wave lengths (neglecting surface tension). In reference *10*, a similar criterion is derived by force balances for the Kelvin-Helmholtz instability. The effect of gravity has been omitted in this work, but surface tension is included. However, growth rates for this type of instability are determined which should be of value for the present work (see Figure 3).

The third possible type of instability is the Rayleigh-Taylor instability. This is an instability of the interface between two fluids of different densities which are stratified or accelerated toward each other. This type of instability arises from the character of the equilibrium of heterogeneous fluids. The mathematical analyses of this instability are also presented in reference 9. They are either of a normal mode or variational type. Studies have been made of both viscous and inviscid fluids, including and neglecting surface tension effects.

The instability in the Rayleigh-Taylor sense neglecting surface tension effects depends only on the relative orientation of the density gradient between the fluids and the acceleration field. For example, horizontal fluid layers in a gravitational field will be stable only if the lower fluid is heavier than the upper; this configuration is unstable (neglecting surface tension) for all wave numbers if the reverse is true. For some unstable situations modes of maximum instability exist and the dependence of the disturbance growth rate on its wave number has been explicitly determined for a number of special cases (*11*).

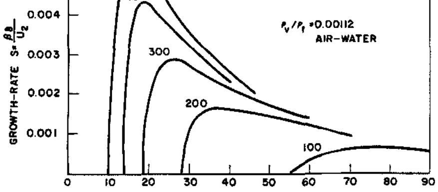

Fig. 3. Growth rate vs. wave length for various Weber numbers (*10*).

The last type of instability capable of influencing two-phase flows is associated with having a single fluid (in contradiction to the Rayleigh-Taylor case) configuration in which the density variation is such that the heavier fluid particles are above the lighter ones in a gravitational field. This is clearly a top-heavy configuration which is potentially unstable. The instability creates a tendency for the fluid to redistribute itself to remedy the weakness in its arrangement. However, the natural tendency of the fluid is inhibited by its own viscosity. In other words, it can be expected that an adverse density gradient which is maintained by viscosity must exceed a certain value before the instability can manifest itself. Considerable study has been given to this type of instability where the density gradient in the fluid results from a temperature gradient. However, similar results will be obtained if the density gradient results from concentration gradients. This type of instability is called the *Bénard instability*. In a horizontal annular condensing flow the film on the upper part of the tube will be subject to an adverse density gradient as a result of both thermal and concentration gradients; the film on the lower part will be in a stable configuration. The flow on the other parts of the tube will have the gravitational force normal to the density gradient and hence will be subject to conventional natural-convection phenomena. Therefore, extreme asymmetry of the flow can result in such a configuration. Note that this discussion of Bénard instability relates to a fluid at rest. If a gas flow is superposed on such a configuration, longitudinal or transverse vortex rolls result. A description of these and the conditions under which they are encountered are given in reference *12*.

The normal mode and variational analyses of the Bénard instability are presented in reference 9. It is shown that a critical value of the Rayleigh number must be exceeded before instability occurs. (The Rayleigh number is the product of the Prandtl and Grashof members where the latter represents the ratio of buoyancy to viscous forces.)

Two other stability analyses (*13, 14*) must be mentioned now because they represent, in essence, an extension and combination of the first two types discussed above to describe more closely the type of phenomena under consideration in the present paper. In particular, in reference *13* consideration is given to the specific case of two continuous, viscous, incompressible fluids in plane motion; one fluid is bounded by a solid wall below and by the second fluid above. The second fluid is unbounded above. The fluid motion is steady and unidirectional, parallel to the interface, and the shear rate in each fluid is uniform. As in the case of the stability of homogeneous fluids, the mathematical analysis is based on small disturbance theory and leads to an eigenvalue problem in a system of two linear ordinary differential equations. In addition to the dimensionless wave number, disturbance phase velocity, and the (inner fluid) Reynolds number that occur in single-fluid studies, the viscosity and density ratios and the Froude and Weber numbers appear as important parameters for nonhomogeneous fluids. The effect of gravity is determined by the Froude number $U_2^2/g\delta$, which represents the ratio of inertia to gravitational forces. The influence of surface tension is determined by the Weber number $\delta\rho_l U_2^2/\sigma$ which essentially denotes the ratio of inertia to surface-tension forces. Single-looped neutral stability maps, similar to those for homogeneous fluids, and disturbance amplifications are presented in reference *13*. In reference *14*, an analysis similar to that of reference *13* is presented, where, however, the only effect of the lighter fluid on the heavier that is included is the shear it imposes on the interface. Thus, the results of reference *14* are for instabilities due to disturbances only in the liquid film. A number of errors in the analysis of reference *13* for this special case are pointed out.

It must be clear that this work also pertains to the plane flow of two fluids under adiabatic conditions with no mass transfer. Therefore, utilization of these results for condensing annular films (as occur in tubes) should be made with caution.

It is important to note that a particular type of instability under given conditions is generally more unstable to one type of disturbance or wave than another. For example, it is well known in hydrodynamic stability theory (*8*) that for incompressible flows two-dimensional disturbances (transverse waves) are more destabilizing in the Tollmien-Schlichting sense than are three-dimensional ones (oblique waves). However, for compressible flows the three-dimensional waves can be more destabilizing. Furthermore, for flows with body forces, longitudinal rather than transverse waves are the more destabilizing. It is perhaps significant in this regard that in the course of experiments on film breakup described in reference *15*, two-dimensional, three-dimensional, and roll-waves were all observed. Further study, therefore, appears warranted on this aspect of the problem; that is, the distinct wave patterns associated with each type of instability should be clearly delineated. In this way it might then be possible to identify the type of flow and instability obtained in experiments, and the most harmful types could be determined.

RELATION OF THE VARIOUS INSTABILITIES TO CONDENSING FLOWS

Now that each possible type of instability has been described, the distinctions among them must be made and their relations to the problem of condensing flows must be established. First of all, note that the Tollmien-Schlichting and Kelvin-Helmholtz instabilities are fundamentally associated with fluids which are in motion; the Rayleigh-Taylor and Bénard instabilities can occur in fluids at rest. The primary factor in the latter two is the body force action. The latter two types would not, therefore, be of consequence in a true zero-gravity environment. The Tollmien-Schlichting and Bénard instabilities can occur in homogeneous fluids as well as in nonhomogeneous ones, *whereas the Kelvin-Helmholtz* and Rayleigh-Taylor instabilities are associated only with the nonhomogeneous fluids.

The Rayleigh-Taylor and Bénard instabilities most likely would not be so important as the other two types for films that have moderate motions because the body forces would be small relative to the inertia forces. The Kelvin-Helmholtz instability would most likely occur before the Tollmien-Schlichting instability because the former has been found to occur even if both fluids are in laminar motion. Near the back end of condenser tubes (and the front end of boiler tubes) the liquid layers meet, and the flow velocity is very small. Under these conditions the Rayleigh-Taylor instability could be important.

In any event, it would seem that determination of the instability most likely to influence a given flow first could be qualitatively obtained by comparing the various existing stability criteria. However, it must be remembered that all the above-described analyses were for plane flows and none included the effects of heat and mass transfer which are an inherent part of the condensing process. Therefore, at best, predictions made on this basis for two-phase annular flows are tenuous. It is, however, interesting to note that even despite the limitations of the existing stability analyses, they can be used to make some of the seemingly anomalous results obtained from the various experiments appear to be reasonable. For example, recall that in references *2* and *3* two changes in wave structure were found. The first change, caused by the energy transmitted across the interface, certainly seems to be the Kelvin-Helmholtz instability; the second change, associated with the liquid flow rate, appears to be the Tollmien-Schlichting instability even though Dukler did not think that the liquid Reynolds number was significant. In reference *6* two different types of surface waves were also found. The first appeared for all liquid flow rates and for shorter wave lengths in consonance with the results of the Kelvin-Helmholtz stability analysis, and the second waves (with long wave lengths) which occurred beyond a certain value of the liquid flow rate or Reynolds number were probably caused by the Tollmien-Schlichting instability. It is possible that Dukler, who did not obtain the first type of waves until a certain amount of energy had been transmitted across the interface, had less disturbances in his experiments than did Knuth. In other words, perhaps Knuth obtained the first waves for all liquid flow rates because his turbulence level was large. These experiments tend to support the conjecture made previously herein that the Kelvin-Helmholtz and Tollmien-Schlichting instabilities are the more important pair for flows with at least moderate liquid velocities and that the Kelvin-Helmholtz instability appears first.

The waviness found in references *4* and *5* after a critical liquid Reynolds number was exceeded must have been caused by the Tollmien-Schlichting instability. The reason no other wave structure corresponding to the Kelvin-Helmholtz instability was reported in those papers is not clear. Either the investigators were not looking for them or the disturbances level was below that required to cause such instability. Note in reference *5* that the flow was that of a single-phase liquid external to a duct, and, therefore, less disturbances were imposed on the liquid film. More careful analysis of the data of those papers might substantiate that the disturbance level was low.

The conclusions of references *4* and *6* that the gas Reynolds number is unimportant as a prime interfacial stability parameter certainly are substantiated by the stability analyses because this parameter does not appear at all. On this basis the results of Laird (*7*) which showed an effect in isolated regions of the gas Reynolds number remain questionable. It may be that the gas-flow turbulence level was responsible for this result. Further discussion of this point will be made subsequently.

In a zero-gravity environment the Rayleigh-Taylor and Bénard instabilities would not exist because there is no body force. It appears that the Kelvin-Helmholtz instability would be the dominant one, at least for film speeds below a rather large value (that is below a large Reynolds number). The reason for this conjecture, again based on the existing analyses, is that this instability can occur even if both fluids are in laminar motion, whereas the Tollmien-Schlichting type occurs when destruction of liquid laminar motion begins. Furthermore, the stability criterion for the Kelvin-Helmholtz type indicates that the flow is unstable for all wave lengths when g goes to zero. Whether, in fact, the situation is as serious as this cannot be determined without more careful study of this type of instability, including such effects as heat, mass transfer, and viscosity.

It is known from hydrodynamic stability studies of the Tollmien-Schlichting type for homogeneous fluids that heat transfer effects can significantly influence stability. For example, cooling of gas layers tends to stabilize them. Since there is considerable heat transfer associated with condensing and boiling films, its effects must be included in future work.

NEW ASPECTS

There is one additional word of caution necessary to be added with regard to the use of existing stability analyses and that is that they all refer to essentially unbounded flow in the direction normal to the surface. In effect, the flows treated theoretically are external ones, whereas for the problem at hand the flows are internal. This geometrical difference may be important by itself; however, there is another aspect which may be of even greater consequence. In an internal flow one part of the liquid layer can produce disturbances to another part. Also in the main (gas) flow there exist such disturbances in internal flows (to a greater degree than in external flows) as turbulence, regular and random sound, temperature spottiness, and the like. These disturbances may cumulate in different and nonlinear ways, and the theory takes no account of these. Therefore, there is some question as to whether stability analyses as described above have any meaning for internal flows. The lack of correlation of hydrodynamic stability theory with transition data for the case of flow in shock tubes has been shown in reference *16*. Laminar internal flows are shown therein to become unstable at Reynolds numbers at least an order of magnitude lower than that predicted by stability theory. This is the opposite of what is meant by flow instability. The question of the validity of hydrodynamic stability analyses for normal internal flows is discussed more fully in reference *16*.

The Kelvin-Helmholtz instability may also be affected by the larger disturbances associated with internal flows. If, as has already been discussed several times, the Kelvin-Helmholtz instability is the first to occur in a given flow, and, furthermore, since it appears that under zero-gravity conditions the flow is unstable for all wave lengths, some means would have to be taken to counteract this instability if it leads to undesirable phenomena. It has been shown that fluid rotation has a stabilizing effect on this type of instability (*9*). Rotation of the fluid should be beneficial in other ways as well. For an annular film (as occurs on the inside surface of a tube) it assures stability in the Rayleigh-Taylor sense, because the body force always acts in the direction of the density increase. Furthermore, the centrifugal force resulting from the fluid rotation will tend to force the heavier liquid particles outward toward the tube surface, and, hence, should reduce the tendency for liquid entrainment.

These two aspects of film instabilities, namely the effect of disturbances external to the liquid film and the influence of fluid rotation, require further study.

FILM BREAKUP

It must be understood that developing valid stability criteria and knowledge of wave amplitudes and growth rates represents only the first important step in identifying and eliminating undesirable phenomena associated with two-phase annular flows. This information indicates the conditions under which interfacial waves occur and what their magnitudes and growth rates are. It may well be that their occurrence cannot be suppressed under required operating conditions. The appearance of the waves, however, does not necessarily mean that serious harmful phenomena will occur. As long as the wave amplitudes remain relatively small, the type of flow may change from, say, annular to fog which leads to some change in the heat transfer and pressure drop, but none of the serious problems, mentioned above, arise until the wave amplitudes become sufficiently large. Therefore, to answer completely the question of whether the waves are harmful or not it would be valuable to be able to predict the time (or length) required for these waves to break up or to attain such amplitudes that they cause flow plugging. A mathematical model is, therefore, proposed herein that will lead to a prediction of the point at which these undesirable conditions will be attained under given conditions. The method depends on the basic data on wave growth, some of which is not yet available. Thus the present formulation is for adiabatic films for which some of this data is available; methods of including the effects of condensing on the instability growth rate and residence time are also discussed.

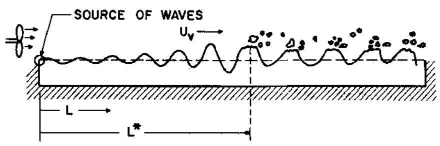

Fig. 4. Wave growth on liquid film.

The growth rate is an important criterion for interface instability. The elevation of the highest amplitude above the undisturbed surface is B. This amplitude grows with the vertical velocity $\dot{B}$; and the rate of growth of this velocity is $\ddot{B}$. Growth rate is now defined as the ratio of $\ddot{B}$ to $\dot{B}$.

The following analysis is of additional value since it indicates clearly what data are lacking to answer the pressing problems conclusively.

Assume a liquid gas interface to be perturbed by disturbances of all wave lengths at a multiplicity of sources and that the disturbance having the maximum positive growth rate dominates the interface. Since the maximum growth rate disturbance can originate at any point along the interface, it will be further assumed that only those formed at the furthest upstream distance will dominate. This assumption precludes the existence of standing waves which is reasonable since none have been observed in previous experiments. This is illustrated in Figure 4 which shows a continuous formation of maximum growth rate waves at $L = 0$. L^* represents the distance at which liquid escapes from the interface and could also represent the distance at which the wave height is sufficient to cause bridging or plugging. Waves of lower growth rate or waves formed at sources where $L > 0$ are eliminated from the picture. This assumption is valid if the maximum growth rate as a function of wave length has a steep maximum. If the curve is flat, then the assumption becomes dubious. A single wave formed at $L = 0$ will appear

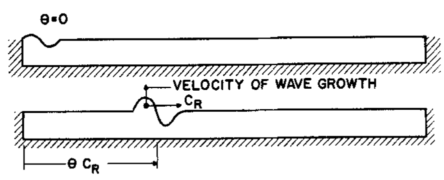

Fig. 5. Propagation of a single wave in a stationary fluid.

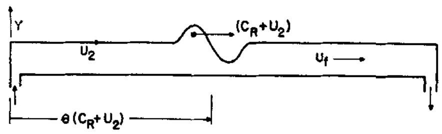

Fig. 6. Propagation of a single wave in a flowing fluid.

chronologically as in Figure 5. Figure 6 shows a wave as it may propagate in reality in a moving interface. It is simpler mathematically (for linear problems such as that of stability) to make a Fourier analysis of such a wave and consider the individual Fourier components. It is, therefore, generally assumed in stability analyses that all the waves are composed of simple harmonic waves conforming with the disturbance stream function

$$\Psi(L, Y, \theta) = f(Y) e^{i\alpha(L - C\theta)} \quad (1)$$

As a result of this assumption all derived quantities, such as the velocities and amplification factor, are harmonic and grow or decay exponentially. Equation (1) also defines the complex wave propagation speed $C \equiv C_R + iC_i$ whose real part C_R is the physical velocity of propagation of this simple wave. The imaginary part, C_i is related directly to the growth rate as will be seen later. The wave speed C is measured with respect to the undisturbed fluid; therefore, the wave velocity relative to a point fixed in space will vary with the velocity of the interface flow U_2 as shown in Figure 6. It is clear from Figure 6 that

$$L = \theta(C_R + U_2) \quad (2)$$

If Equation (2) is applied to a forced convection condensing process in which U_2 and C_R may be functions of length, it is more convenient to write Equation (2) in differential form as

$$dL = d\left[\theta U_2 \left(\frac{C_R}{U_2} + 1\right)\right] = U_2\left(\frac{C_R}{U_2} + 1\right) d\theta + \theta\left(\frac{dC_R}{dU_2} + 1\right) dU_2 \quad (3)$$

For adiabatic fully developed flows the wave propagation velocity and interface velocity will not vary in the downstream direction, so the second term in Equation (3) will vanish. For simplicity, this case will be considered first. Therefore, Equation (3) becomes

$$dL = U_2\left(\frac{C_R}{U_2} + 1\right) d\theta = C_R\left(1 + \frac{U_2}{C_R}\right) d\theta \quad (4)$$

Since the ratio C_R/U_2 appears as a parameter in Equation (4), it is necessary to obtain an estimate of its order of magnitude so that the history of a wave with a maximum growth rate (as illustrated in Figure 6), originating at $L = 0$ and becoming harmful owing to breakup (liquid entrainment) or flow plugging, can be determined. From Feldman's analysis of the Tollmien-Schlichting instability of a plane liquid film the ratio C_R/U_2 is plotted as a function of the property ratios of the two fluids in Figure 7. Figure 7 indicates that this ratio is approximately equal to 0.1. Therefore, Equation (4) can be written as

$$dL = U_2 d\theta \quad (5a)$$

If for other instabilities or flow conditions or configurations $U_2/C_R << 1$, then the other limiting form of Equation (4) is

$$dL = C_R d\theta \quad (5b)$$

The maximum value of this ratio will have to be determined from analysis or experiments for the situation corresponding to annular two-phase condensing flows. In all situations other than the limiting cases [given by Equations (5a) and (5b)], Equation (4) must be used.

It is important to note that the time θ in Equations (4) and (5) depends on the growth rate of the wave, which is related to C_i. This relationship is explicitly determined as follows.

At the interface between the liquid film and gas (or vapor) above it there can be no relative motion between the interface and the liquid; that is

$$V_f \equiv V(b) = \frac{\partial b}{\partial \theta} + (U_2 + C_R)\frac{\partial b}{\partial L}$$

It follows from Equation (1), for the case being considered here $U_2 >> C_R$, that this can be written as

$$V_f = -i\alpha f(b) e^{i\alpha(L - C\theta)} = \frac{\partial b}{\partial \theta} + U_2 \frac{\partial b}{\partial L} \quad (6)$$

It can be shown that by differentiating Equation (6) and assuming that the disturbance velocity component in the L direction is smaller than the wave velocity, that is $f'/C << 1$ (which is physically reasonable), one obtains

$$\frac{d^2 b}{d\theta^2} = \alpha C_i \frac{db}{d\theta} \quad (7)$$

where δ denotes the liquid film height. Then $b = B + \delta$ where B is the wave amplitude. Equation (7) can be written in terms of the wave amplitudes as

$$\frac{\ddot{B}}{\dot{B}} = \alpha C_i \quad (8)$$

upon assuming that the liquid film height δ does not vary with time. Integrating Equation (8) and setting the integration constant equal to zero without loss of generality one obtains

$$\frac{\dot{B}}{B} = \alpha C_i \text{ or } \frac{dB}{B} = \alpha C_i d\theta \quad (9)$$

It is evident from Equation (9) that αC_i is the parameter related to wave growth and, therefore, to the stability of the film.

Combination of Equations (4) and (9) yields

$$\frac{dB}{B} = \frac{\alpha C_i \, dL}{U_2(C_R/U_2 + 1)} \quad (10)$$

If the limiting forms of Equation (5) had been used, the denominator factor in parentheses in Equation (10) would not appear and, for example, in the case ($U_2/C_R << 1$), C_R would appear in place of U_2. In reference *13*, expressions are formulated for $C_i > 0$ (positive growth rates), $C_i = 0$ (neutral stability), C_R/U_2, and α for maximum growth rate waves. These results for Tollmien-Schlichting instability are presented in reference *13* for the case where gravitational and surface tension effects are neglected. Figure 7 is one such set of results. Figures like these together with Equation (10) or its limiting forms are used to determine the breakup or plugging-flow length due to Tollmien-Schlichting instabilities. For other instabilities and configurations, of course, corresponding figures would have to be used, if available, or else determined. For illustrative purposes the results of reference *13* will be used to give an indication of the breakup length associated with Tollmien-Schlichting instabilities; cognizance must be taken, however, that reference *13* pertains specifically only to thin adiabatic liquid films with linear (Couette) velocity profiles. To illustrate the relative dif-

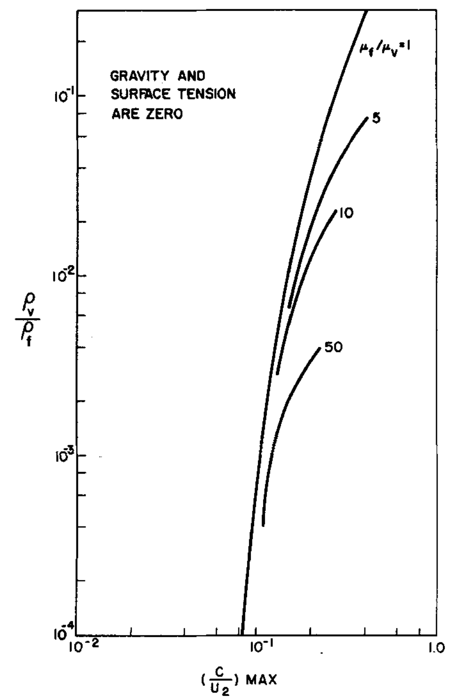

Fig. 7. Maximum wave celerity as a function of property ratios (13).

ference in breakup length for a different instability, the growth rate results of references *10* and *17* for Kelvin-Helmholtz instabilities will then be used.

For adiabatic films to use the data of reference *13* write

$$\frac{\partial C_i}{\partial N_{Rf}} = U_2 F_{max} \tag{11}$$

where

$$F_{max} = f\left(\frac{\rho_v}{\rho_f}, \frac{\mu_f}{\mu_v}\right)$$

Integration of Equation (11) yields

$$C_i = U_2 F_{max} N_{Rf} + K$$

where K is an integration constant. When $C_i = 0$, $N_{Rf} = N_{Rfn}$, where N_{Rfn} is the Reynolds number for neutral stability. Therefore

$$K = -\,U_2 F_{max} N_{Rfn}$$

and

$$C_i = U_2 F_{max}(N_{Rf} - N_{Rfn}) \tag{12}$$

If Equation (12) is substituted into Equation (10), there is obtained

$$\frac{dB}{B} = \frac{\alpha F_{max}(N_{Rf} - N_{Rfn})\, dL}{(C_R/U_2 + 1)} \tag{13}$$

Replace $\alpha = 2\pi/\lambda$ by its dimensionless form (as it appears in reference *13*) $\alpha' = 2\pi\, \delta/\lambda$, so that Equation (13) becomes

$$\frac{dB}{B} = \frac{\alpha' F_{max}(N_{Rf} - N_{Rfn})\, dL}{(C_R/U_2 + 1)\,\delta} \tag{14}$$

To find the breakup or plug flow length, Equation (14) must be integrated. If L^* denotes the point where the waves lead to unfavorable phenomena such as liquid entrainment and flow plugging, then the integration limits are $L = 0$ to $L = L^*$. Similarly, $B = B_1$ to $B = B^*$ where B_1 is the initial wave amplitude and B^* is the amplitude at which either entrainment or plugging occurs. To determine explicitly B^*, the amplitudes for tearing of the interfacial waves can be obtained from experiments, whereas for plugging it is clear that the wave amplitude plus the film thickness must be of the order of one-half the flow passage diameter.

For the case of liquid entrainment it has been shown in references *15, 18,* and *19* that the wave heights approach the magnitude of the average liquid film thickness. Photometric measurements of the wave profile are reported in reference *19*, and it is found that near the point of liquid entrainment $B^*/\delta = 0.81$. Another such figure derived in a similar way (Figure 4*c* of reference *18*) shows that $B^*/\delta = 0.974$.

In reference *18* there is presented the maximum net wave height measurements ($2B$) vs. film thickness. These measurements are limited to conditions for which no liquid is entrained. The point of incipient entrainment occurs when

$$\frac{2B^*}{\delta} = \frac{0.020}{0.010} \text{ or } \frac{B^*}{\delta} = 1$$

Other evidence of this sort is also available. It should be noted that these data were taken for the annular downflow in a vertical tube and hence are more appropriate to the problem under discussion.

To integrate Equation (14) it is also necessary to know the initial wave amplitude B_1. Reference *20* states that $\ln B^*/B$ reaches a constant value for the breakup of moving liquid sheets. (This configuration is again not the same as the one of interest herein but is used merely to illustrate orders of magnitude from existing data.) Reference *21* theoretically predicts values for this ratio from which B_1 can be determined when B^* is known.

For the other undesirable case of flow plugging, assume that flow plugging will occur when $B \approx (D/2) - \delta$. Therefore, for this case the upper integration limit is

$$B_p^* = \frac{D}{2} - \delta$$

which will occur when $L = L_p^*$. If this length is less than that for entrainment L^*, plugging will actually take place in the flow passage. Alternate criteria for plugging may also be developed. For example, if the size of the entrained droplets as computed from reference *22* are of the order of passage dimensions, then plugging can also occur.

Now that the integration limits have been established, Equation (14) can be integrated to give

$$\ln \frac{B^*}{B_1} = \frac{\alpha' F_{max}(N_{Rf} - N_{Rfn})\, L^*}{(C_R/U_2 + 1)\,\delta} \tag{15}$$

Since the concern here is with the maximum growth rate wave, the corresponding values of α' and C_R/U_2 should be inserted in Equation (15). These can be evaluated from figures in reference *13* which, of course, are limited to a specific configuration and instability type as discussed previously. In accordance with these figures F_{max}, α'_{max}, and $(C_R/U_2)_{max}$ are functions of property ratios alone; therefore, for a given combination of fluids at a certain pressure and temperature, these quantities are constants. Also, as was previously indicated, the ratio B^*/B_1 reaches a constant value for liquid film breakup.

Equation (15) can be rewritten as

$$\frac{L^*}{\delta} = \frac{\ln B^*/B_1\,[(C_R/U_2)_{max} + 1]}{\alpha'_{max} F_{max}(N_{Rf} - N_{Rfn})} \tag{16}$$

or

$$\frac{L^*}{\delta} = \frac{E}{(N_{Rf} - N_{Rfn})} \tag{17}$$

where E is a constant for a given set of property values μ_f/μ_v and ρ_v/ρ_f and is explicitly defined as

$$E = \frac{\ln B^*/B_i\,[(C_R/U_2)_{max} + 1]}{\alpha'_{max}\,F_{max}} \tag{18}$$

To show how the breakup length L^* can be found from Equation (17), and to indicate the data required to determine this length for the problem under consideration, use will be made of all existing data even though the configurations and ranges of conditions of the various sources are not consistent. Thus, if air-water systems are considered at atmospheric pressure and temperatures with $\mu_f/\mu_v = 54.1$ and $\rho_v/\rho_f = 0.0012$, then from Figure 7, for Tollmien-Schlichting instability of a thin plane film, $(C_R/U_2)_{max} = 0.1$ and $\alpha'_{max} = 0.6$ from reference *13*. Note that this corresponds to a maximum growth rate wave length-to-film thickness ratio of about 10 which corresponds to the visual experimental observations made in a tube with annular flow reported in reference *23*. From reference *21* $\ln B^*/B_1 = 12$, and from reference *13* $F_{max} = 4 \times 10^{-9}$. Therefore, the magnitude of E is

$$E = \frac{12(0.1 + 1)}{0.6(4 \times 10^{-9})} = 5.5 \times 10^9 \tag{19}$$

This calculation is for film breakup for which liquid drops are entrained. As already pointed out, the integration limits [and therefore $\ln(B^*/B_1)$] would be different for plugging conditions. As a check on the above calculation note that Equation (17) indicates that L^*/δ is a function of N_{R_f}. Data are plotted in terms of these quantities from eight different sources (primarily for annular flows) in Figure 8. Also included in Figure 8 are data for drop formation from rapidly moving liquid sheets (*20*). Figure 8 indicates that the data lie approximately on a 45-deg. line; therefore, the film thickness δ plays no role in the breakup length and $N_{R_{fn}} << N_{R_f}$. Based on the data of Figure 8 $L^*U_2\rho_f/\mu_f$ should approximately equal 1.6×10^6 contrasted to a computed value of $E = 5.5 \times 10^9$ [Equation (19)]. Therefore

$$\frac{L^* U_2 \rho_f}{\mu_f} = 1.6 \times 10^6 \tag{20}$$

(Note: This criterion is analogous to the minimum critical Reynolds number for stability of homogeneous boundary-layer flow.) The value (5.5×10^9) predicted by the method developed herein does not agree with that indicated by experiments (1.6×10^6). The lack of agreement between the predicted and experimental results should not be too disturbing because the predictions were calculated essentially from the work of reference *13* for instability of plane adiabatic thin films in Couette flow. In the experiments the flows were mainly annular with heat transfer in some cases. Furthermore, the analysis of reference *13* which was based on the theory of small perturbations loses accuracy as wave amplitudes become sufficiently large. If the disagreement between predictions and reality persists after more appropriate data for both calculations and experiments have been obtained, the predictions would have to be based on nonlinear theories to account for the fact that the amplitudes near breakup are large. Some study of the growth of waves with large amplitudes has already been made (*24*), and in certain instances it has been found therein that when the wave amplitude has grown sufficiently it remains constant, that is the growth rate goes to zero. Miles (*14*) has improved on some aspects of Feldman's work and presents different growth rate factors. Under conditions appropriate to high-speed films Miles' growth rates are three orders of magnitude greater than Feldman's. Use of Miles' values in the present calculations would then lead to relatively close agreement between the theoretical predictions and experimental data. Since Feldman's calculations are more comprehensive than Miles', his work was used in the present sample calculations. Further work is clearly necessary to resolve the contradictory results just presented.

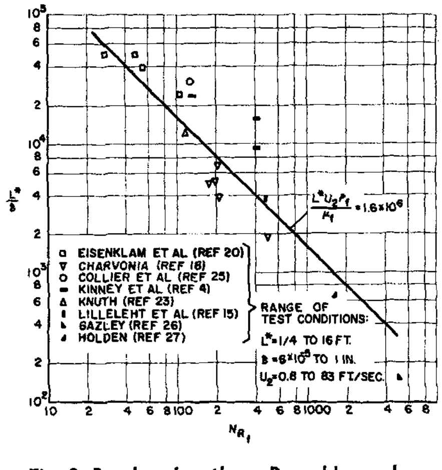

Fig. 8. Breakup length vs. Reynolds number.

To find the film breakup length associated with the Kelvin-Helmholtz instability, Equation (10) will have to be rewritten so that the results of reference *10* which gives the growth rates can be used. To this end Equation (8) is written as

$$\alpha C_i = \frac{\ddot{B}}{\dot{B}} = \frac{\ddot{B}}{B}\,\frac{B}{\dot{B}}$$

which when combined with Equation (9) gives

$$\alpha C_i = \sqrt{\frac{\ddot{B}}{B}} = \beta$$

In reference *10* the growth rate parameter S is given as (see Figure 3)

$$S_{max} = \frac{\beta\,\delta}{U_2}$$

so that

$$\alpha C_i = \frac{S_{max}\,U_2}{\delta}$$

Substitution of this last equation into Equation (10) yields

$$\frac{dB}{B} = \frac{S_{max}}{\delta(C_R/U_2 + 1)}\,dL$$

For the adiabatic case U_2 is independent of L so that integration subject to the condition at $L = 0$, $B = B_1$, gives

$$B = B_1 \exp \frac{S_{max}}{(C_R/U_2 + 1)}\,\frac{L}{\delta}$$

For liquid entrainment

$$\frac{B^*}{B_1} = \exp \frac{S_{max}}{(C_R/U_2 + 1)}\,\frac{L^*}{\delta}$$

In reference *17* for this condition it is found that $\ln B^*/B_1 = 12$; therefore

$$\frac{S_{max}}{(C_R/U_2 + 1)}\,\frac{L^*}{\delta} = 12$$

For the case $C_R/U_2 << 1$ this equation reduces to

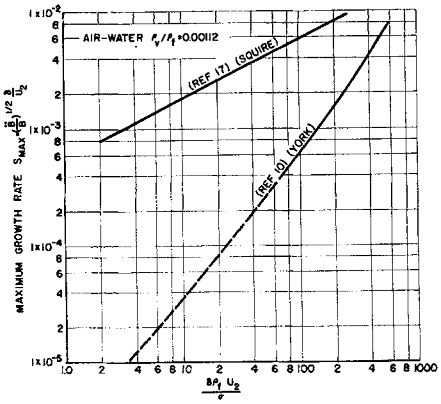

Fig. 9. Maximum growth rate of references *10* and *17* as a function of Weber number.

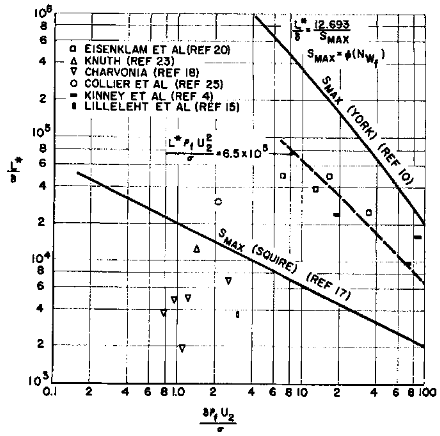

Fig. 10. Breakup length vs. Weber number.

$$S_{\max}\frac{L^*}{\delta} = 12 \tag{21}$$

where $S_{\max}$ is a function of the Weber number. The growth rate parameters taken from reference *10* are plotted in Figure 3. The maximum growth rate parameters obtained from this figure are plotted in Figure 9. The maximum growth rate parameter obtained in reference *17* is also plotted on Figure 9. The results of references *10* and *17* do not agree. Equation (21) with both the maximum growth rate factors of references *10* and *17* used is compared with test data in Figure 10. The data seem to fall between the two curves roughly along a 45-deg. line. From this figure it can tentatively be assumed that for sufficiently high values of Weber number ($N_{W_f} > 10$) the breakup length is

$$\frac{L^*\rho_f U_2^2}{\sigma} = 6.5 \times 10^5 \tag{22}$$

The verification of this equation depends on further experimentation. In a manner consistent with the original premise that the maximum growth rate wave dominates the interfacial breakup, it can be assumed that whichever of Equations (7) or (20) and (22) gives the shorter breakup length is the one to apply. This seems plausible since the greater the growth rate the smaller the breakup length.

Of course, no mutual interference between the Tollmien-Schlichting [Equations (17) or (20)] and the Kelvin-Helmholtz [Equation (22)] instabilities has been assumed in the above development. This assumption certainly has its limitations which have to be established by experiments.

In summary, it can be said that the above development of the film breakup length is general, that is it can be applied to any configuration and type of instability. The specific calculations presented herein were merely for illustrative purposes and to indicate what basic information is required for more realistic results. To apply this analysis to nonadiabatic flows such as are of interest in condensing problems the second term in Equation (3) must be taken into consideration. For practical calculations the relation between U_2 and L can be estimated so that no difficulties, in principle, arise from the inclusion of this term. Thus, only the basic data such as α', C_R/U_2, etc., must be found under nonadiabatic conditions for application of the method to the real case.

CONDITIONS AFTER FILM BREAKUP

Collier and Hewitt (*25*) measured separately the quantity of liquid entrained by the vapor outside the film and the quantity of liquid remaining in the film in a flowing two-phase system. These measurements not only indicate the critical breakup length Reynolds number at the point of incipient entrainment (film breakup) but also show conditions existing after instability occurs. The latter is of considerable practical importance. These conditions can be inferred from Figure 11, where a significant trend in film flow rate with gas flow rate is to be noted beyond the point of instability. Furthermore, the curves level out indicating that beyond some critical conditions the film flow rate no longer increases. An equilibrium film thickness is thus built up consisting probably of a thickness not affected by the waves and an additional thickness to account for the required equilibrium liquid entrainment and deposition rates. A similar situation encountered with a boiling film in forced convection is described in reference *26*.

If this equilibrium film thickness were determined by a liquid Reynolds number, the gas flow rate would not be a significant factor. However, an increase in gas flow rate decreases the film flow rate which implies that possibly a film Weber number is a suitable criterion for determining the conditions for obtaining the equilibrium film thickness. Thus

$$N_{W_{fe}} = \frac{\delta\rho_f U_2^2}{\sigma} = \frac{(\delta\rho_f U_2)\,U_2}{\sigma}$$

where

$$\delta\rho_f U_2 = C_1 m_f$$

and

$$U_2 = C_2 m_g$$

The interface velocity U_2 is related to the gas velocity by means of a slip ratio which is a function of the density ratio of gas to liquid in accordance with reference *27*. Therefore

$$N_{W_{fe}} = \frac{C_1 C_2 m_f m_g}{\sigma}$$

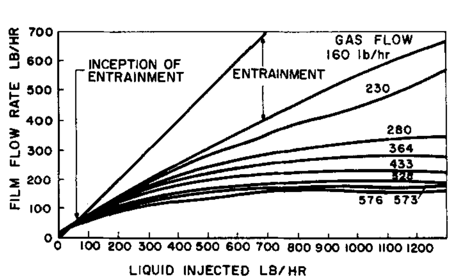

Fig. 11. Result of entrainment (25).

For a constant value of the equilibrium film Weber number, the liquid film flow rate should vary inversely with the gas flow rate. This trend is approximated in Figure 11. However, further work is required for definite substantiation.

SUMMARY OF RESULTS

In this paper it has been pointed out that a number of different physical phenomena can lead to the instability of a liquid film in two-phase flow. With these in mind a better understanding of film breakup is possible, and some anomalous existing results are clarified. It is also indicated that film breakup may result from large scale disturbances in a manner completely different from that described by small disturbance theories.

Instability of the film may not, by itself, be harmful. Therefore, an analysis is made to determine the conditions under which the film would break up or lead to flow plugging in a given configuration. Finally, some evidence is presented that an equilibrium situation is possible after film breakup occurs; the film thickness remains uniform in this case.

The work presented herein then is an attempt to correlate the many different approaches to this complex problem and to indicate what further basic knowledge is required to permit rational design of two-phase flow systems.

NOTATION

b = displacement of the interface
B = amplitude of interface displacement
B^* = wave amplitude where entrainment, or plugging (B_p^*), occurs
$\dot{B}$ = $d^B/d\theta$
$\ddot{B}$ = $d^{2B}/d\theta^2$
B_1 = initial wave amplitude
C = complex wave velocity ($C = C_R + iC_i$)
C_R = real part of propagation velocity
C_i = imaginary part of propagation velocity
d = differential operator
D = tube diameter
N_{Fr} = film Froude number, $U_2^2/g\delta$
f = function to be determined
g = acceleration of gravity
J = Richardson number, $-(gd\rho/dz)/\rho(dU/dz)^2$
L = length
L^* = breakup length for drop detachment
L_p^* = breakup length for plugging
m = mass flow rate
N_{Rf} = Reynolds number, $U_2\rho_f\delta/\mu_f$
N_{Rfn} = film Reynolds number for neutral stability
S_{max} = modulus defined by Equation (24) for maximum growth rate
U = velocity in undisturbed flow
U_2 = interface velocity of liquid film
V_f = vertical velocity component of interface
N_{Wf} = film Weber number, $\frac{\delta\rho_f U_2^2}{\sigma}$
Y = distance normal to the interface

Greek Letters

α = wave number, $2\pi/\lambda$
α' = dimensionless wave number, $2\pi\delta/\lambda$
β = equal to αC_i and $\sqrt{\frac{\ddot{B}}{B}}$
δ = liquid film height or film thickness
θ = time
λ = wave length
μ = shear viscosity
ρ = density
σ = surface tension
Ψ = disturbance stream function

Subscripts

v = vapor
f = liquid
g = gas
i = imaginary
R = real
2 = interface
p = plugging
max = maximum growth rate wave
∞ = great distance outside of boundary layer

LITERATURE CITED

1. Charvonia, D. A., Project Squid, *Tech. Rept. PUR-32-R* (December, 1958).
2. Dukler, A. E., M.S. thesis, University of Delaware, Newark, Delaware (1949).
3. ———, Ph.D. thesis, University of Delaware, Newark, Delaware, (1951).
4. Kinney, G. L., A. E. Abramson, and J. L. Sloop, *Natl. Advisory Comm. Aeronaut. Rept. 1087* (1952).
5. Brauer, H., *VDI Forsch*, **22**, 457 (1956).
6. Knuth, E. L., *Memo No. 20-85* Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California (1953).
7. Laird, A. D. K., Ph.D. thesis, University of California, Berkeley, California (1951).
8. Lin, C. C., "The Theory of Hydrodynamic Stability," Cambridge University Press, England (1955).
9. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability." Clarenden Press, Oxford, England (1961).
10. York, J. L., H. E. Stubbs, and M. R. Tek, *Trans. Am. Soc. Mech. Engrs.*, **75**, 1279 (1953).
11. Reid, W. H., *Proc. Camb. Phil. Soc.*, to be published.
12. Ostrach, S., *Trans. Am. Soc. Mech. Engrs.*, **79**, No. 2, pp. 299-307 (1957).
13. Feldman, S., *J. Fluid Mech.*, **2**, Part 4, pp. 343-370 (1957).
14. Miles, J. W., *ibid.*, **8**, Part 4, pp. 593-611 (1960).
15. Lilleleht, L. U., and T. J. Hanratty, *ibid.*, **11**, 65-81 (1961).
16. Ostrach, S., and P. R. Thornton, *J. Aero-Space Sci.*, **29**, No. 3, (1962).
17. Squire, H. B., *Brit. J. Appl. Phys.*, **4**, 167 (1953).
18. Charvonia, D. A., Am. Soc. Mech. Engrs. Paper No. 61-WA-253 (1961).
19. Warner, C. F., and J. M. Murphy, *Rept. No. F-G1-1*, Jet Propulsion Center, Purdue University, Lafayette, Indiana (March, 1961).
20. Eisenklam, P., N. Dombrowski, and D. Hasson, *Rept. JRL No. 44*, Ministry of Aviation (May, 1959).
21. Weber, C., *ZAMM*, **11**, p. 136 (1931).
22. Lamb, H., "Hydrodynamics," 6 ed., p. 457, Dover Publications, New York (1952).
23. Knuth, E. L., "The Mechanics of Film Cooling," Parts 1 and 2, Jet Propulsion Laboratory (November-December, 1954, January, 1955).
24. Emmons, H., C. Chang, and B. Watson, *J. Fluid Mech.*, **7**, Part 2 (February, 1960).
25. Collier, J. G., and G. F. Hewitt, *Trans. Inst. Chem. Engrs.*, **39**, 127-144 (1961).
26. Tippetts, F. E., Am. Soc. Mech. Engrs. Paper No. 62-WA-161 (1962).
27. Levy, S., Am. Soc. Mech. Engrs. Paper No. 59-HT-15 (1959).
28. Gazley, C., "Co-Current Gas-Liquid Flow III. Interfacial Shear and Stability," Heat Transfer and Fluid Mechanics Institute (1949).
29. Holden, E. K., M.Ch.E. thesis, University of Delaware, Newark, Delaware (1948).

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